

# Some Constructions of Divisible Designs from Laguerre Geometries

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## Abstract

In the nineties, A.G. Spera introduced a construction principle for divisible designs. Using this method, we get series of divisible designs from finite Laguerre geometries. We show a close connection between some of these divisible designs and divisible designs whose construction was based on a conic in a plane of a 3-dimensional projective space.

Keywords: Divisible Designs, Laguerre Geometries, Dual Numbers, Automorphism Groups

## 1 Introduction

One interesting sort of designs is that of divisible designs. In 1992, A. G. Spera introduced a method to construct these designs by  $t - R$ -homogeneous ( $t - R$ -transitive)  $R$ -permutation groups ([9]). Here,  $R$  denotes an equivalence relation on the elements of a finite set. Once we have such an  $R$ -permutation group acting on a finite set  $X$ , the main problem is the calculation of the parameters of the divisible design. Especially the determination of the order of the stabilizer of a chosen base block in the  $R$ -permutation group, which is needed for the calculation of the parameter  $\lambda$ , is often not trivial. For that purpose, we have to obtain suitable conditions on the construction. There are already several known examples of constructions using Spera's construction principle ([9], [8], [4],[3]). In 1999, C. Cerroni and R.-H. Schulz gave one such construction starting from a conic in a plane of the 3-dimensional projective space  $\text{PG}(3, q)$  ([4]).

In this paper, we construct several series of new 3-divisible designs again using this method but starting from a Laguerre geometry  $\Sigma(\text{GF}(q), \mathbb{D}(\text{GF}(q)))$  with  $\mathbb{D}(\text{GF}(q))$  being the ring of dual numbers over the finite field  $\text{GF}(q)$ .

We show a close connection between some of these designs and those constructed by Cerroni and Schulz. We will see that the initial situations of both constructions are mutually dual and that the parameters of our first series of 3-divisible designs of Theorem 3.1 are equal to the parameters of the series of 3-divisible designs by Cerroni and Schulz (cf. Theorem 2.7) mentioned above. The other series of Theorem 3.1 seem to be new.

## 2 Preliminary terms, definitions and results

### 2.1 About Divisible Designs

Let  $X$  be a finite set with an equivalence relation  $R$  on its elements. We denote by  $[x]$  the  $R$ -equivalence class containing  $x \in X$  and define  $S := \{[x] | x \in X\}$ .

**Definition 2.1** A subset  $Y$  of  $X$  is called  *$R$ -transversal*, if  $|Y \cap [x]| \leq 1$  for all  $x \in X$ .

**Definition 2.2 (Definition of divisible designs)** Let  $t, s, k, \lambda_t$  be positive integers with  $t \leq k < v = |X|$ . A triple  $\mathcal{D} = (X, \mathcal{B}, S)$  is called  *$t - (s, k, \lambda_t)$ -divisible design* (or  *$t - (s, k, \lambda_t)$ -DD*) if

- (1)  $\mathcal{B}$  is a set of  $R$ -transversal subsets of  $X$  with  $|B| = k$  for all  $B \in \mathcal{B}$ ;
- (2)  $|[x]| = s$  for all  $x \in X$ ;
- (3) for every  $R$ -transversal  $t$ -subset  $Y$  of  $X$  there exist exactly  $\lambda_t$  elements of  $\mathcal{B}$  containing  $Y$ .

The elements of  $X$  are called *points*, those of  $\mathcal{B}$  *blocks*, and the elements of  $S$  *point classes*.

In this paper, we always suppose that every divisible design is *simple*, that means that there exist no repeated blocks. Note that a  $t$ -divisible design is also a  $(t - 1)$ -divisible design with  $\lambda_{t-1} = \lambda_t(v - st + s)(k - t + 1)^{-1}$ . We shall use this observation and consider as well the 2-divisible designs arising from the 3-divisible designs to be constructed below.

One way of constructing divisible designs is given by the following proposition of A.G. Spera.

### 2.1.1 SPERA's construction principle

**Definition 2.3** Let  $G$  be a group acting on the set  $X$  and  $R$  an equivalence relation on  $X$  which is  $G$ -invariant, that is,

$$xRy \implies x^g R y^g \text{ (for all } g \in G, x, y \in X);$$

then  $\Lambda = (G, X, R)$  is called an  $R$ -group. (The group  $G$  induces a permutation group on  $X$ , but not necessarily faithfully.)

**Definition 2.4**  $\Lambda$  is called  $t$ - $R$ -transitive if for any two  $R$ -transversal  $t$ -tuples  $(x_1, \dots, x_t)$  and  $(y_1, \dots, y_t)$  of elements of  $X$  there exists an element  $g$  of  $G$  such that  $y_i = x_i^g$  for  $i = 1, 2, \dots, t$ .

**Proposition 2.5 (A.G. Spera, [9])** Let  $\Lambda = (G, X, R)$  be a finite  $t$ - $R$ -transitive  $R$ -group, and let  $B$  be an  $R$ -transversal subset of  $X$  with  $t \leq k := |B| < v := |X|$ , then the incidence structure  $D(\Lambda, B) = (X, B^G, S)$  for  $B^G = \{B^g \mid g \in G\}$  is a  $t - (s, k, \lambda_t)$ -divisible design with  $s = |[x]|$  for some  $x \in X$ ,  $k = |B|$ ,

$$b = \frac{|G|}{|G_B|} \text{ and } \lambda_t = |G| \binom{k}{t} \left( |G_B| \binom{vs^{-1}}{t} s^t \right)^{-1},$$

where  $G_B$  denotes the setwise stabiliser of  $B$  and  $b$  the number of blocks of  $D(\Lambda, B)$ . Moreover,  $G$  induces a point- and block-transitive automorphism group of  $D(\Lambda, B)$ .

**Remark 2.6** To construct divisible designs by using Spera's proposition, we need a finite set  $X$  with an equivalence relation  $R$  on its elements and a finite  $t$ - $R$ -transitive  $R$ -permutation group acting on this set. Then, we have to choose a so called 'base block' and calculate the parameters.

By using Spera's proposition, C. Cerroni and R.-H. Schulz constructed the following series of divisible designs [4].

**Theorem 2.7 (Cerroni, Schulz)** Let  $q = p^n$ , where  $p$  is a prime, and let  $n, i \in \mathbb{N}$  with  $i|n$ . If  $q$  is odd, there exists a  $3 - (q, p^i + 1, 1)$  - DD with  $q^2 + q$  points, having as a point- and block-transitive automorphism group  $T\tilde{G}$  with  $\tilde{G} \cong \text{GO}(3, q)$  and  $T$  the translation group of  $\text{AG}(3, q)$ .

By starting from  $\text{PGO}(3, q)$ , which acts 3-transitively on a given conic in a plane of the 3-dimensional projective space  $\text{PG}(3, q)$ , Cerroni and Schulz constructed a 3- $R$ -transitive  $R$ -permutation group ( $R$  denotes the parallelism

relation) of a finite set of affine planes in the corresponding<sup>1</sup> 3-dimensional affine space. After choosing a base block, they used Spera's proposition to construct these divisible designs.

The whole construction and the proof can be found in [4]. In this paper, we will describe the idea of the construction in part 4 in order to compare it with our construction below.

## 2.2 A Laguerre Geometry over the finite field $\text{GF}(q)$

**Definition 2.8** A *dual number over the finite field  $\text{GF}(q)$*  is an ordered pair  $(a, b)$  with  $a, b \in \text{GF}(q)$  and with the following properties.

Two dual numbers are equal if their components are equal. The rules for addition and multiplication are:

- (i)  $(a, b) + (a', b') := (a + a', b + b')$
- (ii)  $(a, b) \cdot (a', b') := (aa', ab' + ba')$ .

A dual number  $(a, b)$  can also be represented either by a matrix  $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$  with  $a, b \in \text{GF}(q)$  or in the following form:  $a + b\epsilon$  where  $\epsilon$  is any chosen element satisfying  $\epsilon^2 = 0$ , for instance  $\epsilon = (0, 1)$ . The rules for addition and multiplication correspond to those for matrices.

**Remark 2.9** As a subring of the ring of matrices, the set of dual numbers with the given addition and multiplication is a ring with 1. We denote this ring by  $\mathbb{D}(\text{GF}(q))$ .

In analogy to Benz ([1], p.24), we have

**Proposition 2.10**  $\mathbb{D}(\text{GF}(q))$  is a commutative local ring.

The only maximal ideal  $N$  contains all non-invertible elements of  $\mathbb{D}(\text{GF}(q))$ . For any ring  $R$ , we define  $R^*$  as the multiplicative group of all invertible elements. Here  $\mathcal{R} := \mathbb{D}(\text{GF}(q))^* = \mathbb{D}(\text{GF}(q)) \setminus N$ .

**Definition 2.11 (Laguerre algebra)** For  $K$  a field, a  $K$ -algebra  $A$  is called a *Laguerre algebra* provided there exists a two-sided ideal  $M$  of  $A$  with  $A^* = A \setminus M$  and  $A = K \oplus M$ .

$\mathbb{D}(\text{GF}(q))$  is commutative, hence the above defined ideal  $N$  is two-sided. Furthermore,  $\mathbb{D}(\text{GF}(q))$  is the direct sum of the embedded field  $\text{GF}(q)$  with

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<sup>1</sup>The 3-dimensional affine space whose ideal plane is the plane considered above containing the given conic.

$N$ , where  $\text{GF}(q)$  is identified with the set of diagonal matrices  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$  (or the set of pairs  $(a, 0)$  or  $a + 0\epsilon$  with  $a \in \text{GF}(q)$ , depending on the manner of representation), hence we get the well known result

**Proposition 2.12**  $\mathbb{D}(\text{GF}(q))$  is a Laguerre algebra.

**Definition 2.13** We define the *projective line*  $\mathbb{P}(\mathbb{D}(\text{GF}(q)))$  over  $\mathbb{D}(\text{GF}(q))$  as the set of all equivalence classes of admissible pairs. Here, we call a pair  $(x_1, x_2)$ ,  $x_1, x_2 \in \mathbb{D}(\text{GF}(q))$  an *admissible pair over*  $\mathbb{D}(\text{GF}(q))$  if at least one element is invertible. Two admissible pairs  $(x_1, x_2), (y_1, y_2)$  are called *equivalent* if there exists an element  $r \in \mathcal{R}$  such that  $x_i = ry_i$ ,  $i = 1, 2$ . We call the elements of  $\mathbb{P}(\mathbb{D}(\text{GF}(q)))$  *points*. Since  $\mathbb{D}(\text{GF}(q))$  is a local ring, this definition of the projective line over  $\mathbb{D}(\text{GF}(q))$  is equivalent to that given by Herzer [5] on p. 785.

**Definition 2.14** Two points  $P, Q \in \mathbb{P}(\mathbb{D}(\text{GF}(q)))$  with  $P = \mathcal{R}(p_1, p_2), Q = \mathcal{R}(q_1, q_2)$ ,  $p_i, q_i \in \mathbb{D}(\text{GF}(q))$ ,  $i = 1, 2$  are called *parallel* if

$$p_1q_2 - q_1p_2 = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \notin \mathcal{R}.$$

In analogy to the more general definition of chain geometry by Benz in ([1], p. 94), we define, in this paper, the chain geometry  $\Sigma(K, D)$  with  $K := \text{GF}(q), D := \mathbb{D}(\text{GF}(q))$  as an incidence structure whose points are the elements of  $\mathbb{P}(D)$  and whose blocks (chains) are the images of  $\mathbb{P}(K)$  under the projective group of  $\mathbb{P}(D)$  (cf. Def. 2.19 and [5], p. 790). Since  $D$  is a Laguerre algebra, the following holds:

**Proposition 2.15** ([5])  $\Sigma(K, D)$  is a so called Laguerre geometry, i.e., the parallelism relation is an equivalence relation on  $\mathbb{P}(D)$  and every chain of  $\Sigma(K, D)$  meets every parallel class of points.

Now consider the chain geometry  $\Sigma(K, D)$  whose points are the elements of the projective line over  $D$ . We can partition the points of  $\mathbb{P}(D)$  into *proper* and *improper* points depending on the invertibility of the second component or, equivalently, on the parallelism to the point  $\mathcal{R}(1, 0)$ . Every proper point can be represented as  $\mathcal{R}(p, 1)$ ,  $p \in D$  and every improper point as  $\mathcal{R}(1, \delta\epsilon)$ ,  $\delta \in K$ .

**Proposition 2.16** (i) Let  $P, Q \in \mathbb{P}(D)$  be two proper points with  $P = \mathcal{R}(p_1 + p_2\epsilon, 1)$  and  $Q = \mathcal{R}(q_1 + q_2\epsilon, 1)$ . They are parallel iff  $p_1 = q_1$ .

(ii) Improper points are always parallel.

(iii) A proper point is never parallel to an improper point.

The easy proof of this uses

$$P \parallel Q \Leftrightarrow \begin{vmatrix} p_1+p_2\epsilon & 1 \\ q_1+q_2\epsilon & 1 \end{vmatrix} = p_1 - q_1 + (p_2 - q_2)\epsilon \notin \mathcal{R} \Leftrightarrow p_1 - q_1 = 0.$$

**Remark 2.17** By the parallelism relation, we get  $q + 1$  equivalence classes with  $q$  elements each:

$$\{\mathcal{R}(x + b\epsilon, 1) | b \in K\}, \text{ with } x \in K \text{ and } [\mathcal{R}(1, 0)] = \{\mathcal{R}(1, \delta\epsilon) | \delta \in K\}.$$

**Remark 2.18** We can embed the projective line over  $K$  into  $\mathbb{P}(D)$ . The elements of  $\mathbb{P}(K)$  form the following transversal subset of  $\mathbb{P}(D)$ :

$$\tilde{K} := \{\mathcal{R}(p_1 + 0\epsilon, 1) | p_1 \in K\} \cup \mathcal{R}(1, 0).$$

**Definition 2.19** One defines the projective group of  $\mathbb{P}(D)$  as the group of all regular  $2 \times 2$  matrices with entries in  $\mathbb{D}(k)$  factorised by the subgroup  $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} | r \in \mathcal{R} \right\}$ . We denote it by  $\Gamma(D)$ . (cf. [1])

**Proposition 2.20** ([1])  $\Gamma(D)$  acts sharply 3- $R$ -transitively on the point set of  $\mathbb{P}(D)$  and preserves parallelism.

By Remark 2.6, we are now able to construct a divisible design using Proposition 2.5.

### 3 Construction of divisible designs from a Laguerre geometry

**Theorem 3.1** Let  $n, i \in \mathbb{N}$  with  $i | n$  and let  $q = p^n$ , where  $p$  is a prime. Then, there exist  $3 - (q, k, \lambda_3)$ -divisible designs, each with  $q(q + 1)$  points with the parameters  $k$  and  $\lambda_3$  given in Table 1, where  $p$  and  $i$  are subject to the conditions given there. These 3-divisible designs admit  $\Gamma(\mathbb{D}(\text{GF}(q)))$  as a point- and block-transitive automorphism group. The same holds for the corresponding 2-divisible designs.

*Proof of Theorem 3.1:* Consider the chain geometry  $\Sigma(\text{GF}(q), \mathbb{D}(\text{GF}(q)))$ ,  $q = p^n$  where  $p$  is a prime. We use the same notation as above. By Prop. 2.20, we have, with  $\Gamma(D)$ , a 3- $R$ -transitive  $R$ -permutation group acting on the point set which consists of the  $q^2$  proper points and  $q$  improper points. They are divided into  $q + 1$  parallel classes with  $q$  elements each, giving the points and the point classes of a DD. By using the sharp 3- $R$ -transitivity

Table 1:

No. <sup>2</sup>	$k$	$\lambda_3$	Conditions
(i)	$p^i + 1$	1	
(ii)	$p^i$	$p^i - 2$	$p^i > 2$
(iii)	$p^i - 1$	$\frac{1}{2}(p^i - 2)(p^i - 3)$	$p^i > 3$
(iv)	$p^i - 2$	$\frac{1}{6}(p^i - 2)(p^i - 3)(p^i - 4)$	$p^i > 4$
(v)	$p^i - 3$	$\frac{1}{4}(p^i - 3)(p^i - 4)(p^i - 5)$	$p^i > 7$
	4	6	
	$p^i - 3$	$\frac{1}{24}(p^i - 3)(p^i - 4)(p^i - 5)$	$p = 3$ and $p^i > 5$
	4	1	
	$p^i - 3$	$\frac{1}{8}(p^i - 3)(p^i - 4)(p^i - 5)$	$p > 3$ and $p^i > 5$
	4	3	
	$p^i - 3$	$\frac{1}{12}(p^i - 3)(p^i - 4)(p^i - 5)$	$p^i \equiv 1 \pmod{3}$ and $p^i > 5$
	4	2	

and determining the order of the orbit of a transversal triple, we obtain  $|\Gamma(D)| = q^4(q^2 - 1)$ . Now we determine the order of the stabiliser of the considered base block which we choose for the different cases.

Let  $i \in \mathbb{N}$  with  $i|n$  where  $q = p^n$ .

- (i) For  $L := \text{GF}(p^i)$ , we embed the projective line over  $L$

$$\mathbb{P}(L) = \text{PG}(1, p^i) =: B$$

into the projective line over  $D$  and define it as our base block.

Notice that a projectivity of  $\Gamma(D)$ , which maps three distinct points of  $B$  onto points of  $B$ , belongs to  $\Gamma(L)$  (cf. [5], Prop. 2.3.1, p.790).  $\Gamma(L)$  acts sharply 3-transitively on  $\text{PG}(1, p^i)$  and therefore (regarded as a subset of  $\Gamma(D)$ ) on  $B$ , too. Hence, the order of the stabiliser of  $B$  is  $|\Gamma(L)| = p^i(p^{2i} - 1)$ .

In  $\Sigma(L, D)$ , three mutually nonparallel points are incident with exactly one chain (cf. [1], Theorem 1.1, p. 95), and the chains are precisely the blocks of our divisible design. By Prop. 2.5, we get a  $3 - (s, k, \lambda_3)$ -DD with  $s = q$ ,  $k := |B| = p^i + 1$  and  $\lambda_3 = 1$ .

This is also a  $2 - (q, p^i + 1, \frac{q(q-1)}{p^i-1})$ -divisible design.

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<sup>2</sup>The number refers to the corresponding part of the proof below.

By removing a set  $M \subset B$ , we define  $B' := B \setminus M$  as our base block to construct the DD's of the cases (ii)-(v). The block  $B'$  should contain at least three points, therefore,  $p^i$  has to be big enough.

The stabiliser of  $B'$  in  $\Gamma(D)$  has to be in  $\Gamma(L)$  (see above) and thus to fix  $B$  and hence also  $M$  setwise. Therefore,

$$\Gamma(D)_{B'} = \Gamma(L)_M.$$

Notice that, vice versa, the stabiliser of  $M$  in  $\Gamma(D)$  is equal to the stabiliser of  $B'$  in  $\Gamma(D)$  iff  $M$  consists of at least three elements.

- (ii) Let  $B' := B \setminus \{\mathcal{R}(1, 0)\}$ . This is an  $L$ -chain minus one point.  $\Gamma(L)$  acts sharply 3-transitively on  $B$ , hence  $|\Gamma(L)| = |\Gamma(L)_{\mathcal{R}(1,0)}|(p^i + 1)$ , and therefore  $|\Gamma(L)_{B'}| = p^i(p^i - 1)$ .

By Proposition 2.5, we get a  $3 - (s, k, \lambda_3)$ -divisible design with  $s = q$ ,

$$k := |B'| = p^i \text{ and } \lambda_3 = \frac{|\Gamma|}{|\Gamma_{B'}|} \binom{p^i}{3} / \left[ \binom{\frac{q^2+q}{q}}{3} q^3 \right] = p^i - 2.$$

This is also a  $2 - (q, p^i, q(q-1))$ -divisible design.

- (iii) Let  $B' := B \setminus \{\mathcal{R}(1, 0), \mathcal{R}(0, 1)\}$ .

From the sharp 3-transitivity of  $\Gamma(L)$  and the number of possible permutations of the elements of  $M$ , we know  $|\Gamma(L)_{\{\mathcal{R}(1,0), \mathcal{R}(0,1)\}}| = 2(p^i - 1)$ . Therefore, by Proposition 2.5, we get a  $3 - (s, k, \lambda_3)$ -divisible design with  $s = q$ ,  $k := |B'| = p^i - 1$  and  $\lambda_3 = \frac{1}{2}(p^i - 2)(p^i - 3)$ .

This is also a  $2 - (q, p^i - 1, \frac{1}{2}(p^i - 2)q(q-1))$ -divisible design.

- (iv) Let  $B' := B \setminus \{\mathcal{R}(1, 0), \mathcal{R}(0, 1), \mathcal{R}(1, 1)\}$ .

Similar to (iii), we can conclude  $|\Gamma(L)_{\{\mathcal{R}(1,0), \mathcal{R}(0,1), \mathcal{R}(1,1)\}}| = 6$  from the sharp 3-transitivity of  $\Gamma(L)$  and the fact that there exist  $3! = 6$  possible permutations of the elements of  $M$ . Now, we get a  $3 - (s, k, \lambda_3)$ -divisible design with  $s = q$ ,  $k := |B'| = p^i - 2$  and  $\lambda_3 = \frac{1}{6}(p^i - 2)(p^i - 3)(p^i - 4)$ .

This is also a  $2 - (q, p^i - 2, \frac{1}{6}(p^i - 2)(p^i - 3)q(q-1))$ -divisible design.

- (v) Let  $B' := B \setminus \{\mathcal{R}(1, 0), \mathcal{R}(0, 1), \mathcal{R}(1, 1), \mathcal{R}(x, 1)\}$ , with  $x \in L \setminus \{0, 1\}$ .

In analogy to the cases above, the stabiliser of  $B'$  in  $\Gamma(L)$  corresponds to the stabiliser of  $M$  in  $\Gamma(L)$ . Two 4-tuples of points are projectively equivalent iff their cross-ratios are equal. The four points of  $M$  allow 24 permutations, but they determine only the following six cross-ratios:

$$x, 1/x, 1-x, 1/(1-x), (x-1)/x, x/(x-1) \quad (*)$$



In any case, the cross-ratio of the four points is invariant under a projective group of order 4 isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  ([6], p.119/120).

- (a) If all six values of  $(*)$  are different, then the stabiliser of  $B'$  consists only of these four elements since  $24 = 6 \cdot 4$ . This case occurs if the four points form neither a harmonic nor an equianharmonic quadruple.

If  $p$  is even, no harmonic quadruple exists, so  $x$  can be any element of  $\text{GF}(p^i) \setminus \{0, 1\}$  which is not a solution of  $x^2 - x + 1 = 0$ . We can choose a suitable point  $\mathcal{R}(x, 1)$  since there exist at most two solutions of this equation and since  $p^i > 5$  is assumed.

To get such a point if  $p$  is odd, we have to assume that  $x$  is neither an element of  $\{0, 1, -1, 1/2, 2\}$  nor a solution of  $x^2 - x + 1 = 0$ ; hence,  $p^i > 7$  is sufficient.

- (b) If at least two of the values of  $(*)$  are equal, the four points form a harmonic quadruple if the values of the cross-ratios are  $\{-1, 1/2, 2\}$  or an equianharmonic one if  $x = 1/(1 - x)$  (or equivalently  $x = (x - 1)/x$ ) or both, which is called superharmonic by Hirschfeld ([6]) and which occurs if and only if  $p = 3$ . In this case, the stabiliser of  $B'$  is the symmetric group  $S_4$  of order 24.

A harmonic quadruple where  $p > 3$  is stabilized by the dihedral group  $D_4$  of order 8. Equianharmonic quadruples exist precisely when  $p^i \equiv 1 \pmod{3}$  and their stabiliser is the alternating group  $A_4$  of order 12 ([6], p.121).

Let  $B'$  be a 4-subset of  $B$ , then we get the same groups as stabilisers as above. By Proposition 2.5, we get the divisible designs of Theorem 3.1.1, part (v).  $\square$

**Remark 3.2** A.G. Spera constructed divisible designs from a finite local  $K$ -algebra  $A$  with  $K = \text{GF}(q)$  and  $J$  its Jacobson radical (with  $|A| = q^n$ ,  $|J| = q^j$ ,  $n, j \in \mathbb{N}$ ). In the special case  $K \cong A/J$ , where  $A$  is a Laguerre algebra, he obtained a transversal  $3 - (q^j, q + 1, 1)$ -DD as in case (i) (cf. [10]).

## 4 Comparing both constructions

In the introduction, we already mentioned a connection between our construction and that of Cerroni and Schulz [4]. Since this connection is not obvious, we mention another representation of the Laguerre geometry  $\Sigma(\text{GF}(q), \mathbb{D}(\text{GF}(q)))$ .

After that, we give a short description of Cerroni and Schulz's construction which will show the duality.

Similar to Blaschke's Cylinder-Model ([2], [1]) in the real 3-space, it is possible to embed  $\Sigma(K, D)$  in a 3-dimensional projective space  $\Psi$  (cf. [7]). By using the more general case showed by Hotje [7], we can identify the elements of  $\mathbb{P}(D)$  with the elements of a quadratic cone  $\mathcal{O}$ , except its vertex  $E$ , in the 3-dimensional projective space  $\Psi$ . Similar to Blaschke's Cylinder-Model, two points are parallel iff they lie on the same generator<sup>3</sup> of  $\mathcal{O}$  [7]. Consider a plane in  $\Psi$ , whose intersection with  $\mathcal{O}$  is exactly the point  $E$ , then all points of the cylinder  $Z := \mathcal{O} \setminus E$  are affine points of the 3-dimensional affine space whose ideal plane is the plane considered above. Such a plane exists since no finite field is quadratically closed.  $Z$  consists of  $q + 1$  lines (generators) each containing  $q$  points and intersecting the ideal plane in  $E$ . Each line contains precisely one parallel class of points.

Now, keeping this in mind, we turn to Cerroni and Schulz's construction [4]. Consider a conic  $O$  in the ideal plane  $E'$  of the 3-dimensional affine space  $\text{AG}(3, q)$ . There is a unique tangent at each point of  $O$  which determines precisely one parallel class of affine planes. Planes of the same parallel class all intersect  $E'$  in the appropriate tangent. In this way, one gets  $q + 1$  parallel classes each consisting of  $q$  planes which are the points of the constructed divisible designs. By dualising  $\Psi$ , we obtain the situation of this construction in which the plane  $E'$  is dual to the point  $E$  and the planes of one parallel class correspond to the points of one generator of  $Z$ , respectively. The series of divisible designs of Theorem 3.1, part (i) possess the same parameters as the series constructed by Cerroni and Schulz (cf. Theorem 2.7). They seem to be mutually dual, whereas the other series of divisible designs of Theorem 3.1, arising from different base blocks, seem to be new.

## References

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<sup>3</sup>A generator of  $\mathcal{O}$  is a line which is completely contained in  $\mathcal{O}$ .

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